A spectrum result on maximal partial ovoids of the generalized quadrangle $\mathcal{Q}(4,q)$, q odd

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Abstract. In this article, we prove a spectrum result on maximal partial ovoids of the generalized quadrangle $\mathcal{Q}(4,q)$, q odd, i.e. for every integer k in the interval $[a,b]$, where $a \approx \frac{3}{5}q^2$ and $b \approx \frac{9}{10}q^2$, there exists a maximal partial ovoid of $\mathcal{Q}(4,q)$, q odd, of size k. Since the generalized quadrangle $\mathcal{W}(q)$ defined by a symplectic polarity of $\mathrm{PG}(3,q)$ is isomorphic to the dual of the generalized quadrangle $\mathcal{Q}(4,q)$, the same result is obtained for maximal partial spreads of $\mathcal{W}(q)$, q odd. This article concludes a series of articles on spectrum results on maximal partial ovoids of $\mathcal{Q}(4,q)$, on spectrum results on maximal partial spreads of $\mathcal{W}(q)$, on spectrum results on maximal partial 1-systems of $\mathcal{Q}^+(5,q)$, and on spectrum results on minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$. We conclude this article with the tables summarizing the results.

1. Introduction

A *generalized quadrangle* Γ is an incidence structure consisting of points and lines such that:

(a) any two distinct points are on at most one line,
(b) every line is incident with $s+1$ points and every point is incident with $t+1$ lines,
(c) if a point P is not incident with the line ℓ, then there is exactly one line through P intersecting ℓ.

The generalized quadrangle Γ is said to have *order* (s,t) or *order* s if $s=t$; the number of points of Γ is $(s+1)(st+1)$ and the number of lines is $(t+1)(st+1)$. Dualizing Γ, we get a generalized quadrangle of order (t,s). For more information on generalized quadrangles, we refer to [**5**].

An *ovoid* $\mathcal{O}$ of a generalized quadrangle Γ is a set of points such that every line of Γ contains exactly one point of $\mathcal{O}$. A *partial ovoid* $\mathcal{O}$ of Γ is a set of points such that every line of Γ contains at most one point of $\mathcal{O}$, and $\mathcal{O}$ is called *maximal* if it is not contained in a larger partial ovoid.

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Let $\mathcal{Q}(4,q)$ be a non-singular parabolic quadric in the projective space $\mathrm{PG}(4,q)$; the set of points and the set of lines of $\mathcal{Q}(4,q)$ form a generalized quadrangle of order q. The points of $\mathrm{PG}(3,q)$ and the self-polar lines of a symplectic polarity σ of $\mathrm{PG}(3,q)$ form the generalized quadrangle $\mathcal{W}(q)$ of order q, which is isomorphic to the dual of $\mathcal{Q}(4,q)$. The size of an ovoid of a generalized quadrangle Γ of order (s,t) is $st+1$, hence an ovoid of $\mathcal{Q}(4,q)$ has size q^2+1.

There is an interest for the size of maximal partial ovoids of $\mathcal{Q}(4,q)$; in [**2**], the authors prove that the size of the smallest partial ovoid of $\mathcal{Q}(4,q)$ is $q+1$ if q is even and at least $1,419q$ if q is odd, while in [**1**] the authors prove that the size of the largest maximal partial ovoid, different from an ovoid, is q^2-q+1 if q is even, and in [**3**], that it is at most q^2-3 when q is odd and not a prime. In [**6**], the authors prove a spectrum result for the size of maximal partial ovoids of $\mathcal{Q}(4,q)$, q even; that is, they find an interval $[a,b]$, where $a \approx q^2/10$ and $b \approx \frac{9}{10}q^2$, such that for every integer $k \in [a,b]$, there exists a maximal partial ovoid of $Q(4,q)$, q even, of size k. The aim of this article is to prove a similar result for $Q(4,q)$, q odd.

2. The technique

We apply the idea behind the construction presented in [**9**] which is used to find minimal blocking sets in $\mathrm{PG}(2,q^2)$. They consider a particular minimal blocking set in the plane $\mathrm{PG}(2,q^2)$, namely the Hermitian curve $\mathcal{H}(2,q^2)$, then replace q of the points lying on a secant line ℓ by the point $\ell^\perp$. They obtain in this way a new minimal blocking set of the plane, but of a smaller size. It is clear that in this construction the polarity of the Hermitian curve plays an important role, and so it does also in ours.

The quadric $\mathcal{Q}(4,q)$, q odd, induces a polarity $\perp$ in $\mathrm{PG}(4,q)$ and we will widely use that polarity. The points of $\mathcal{Q}(4,q)$ are called *singular*; if two singular points are joined by a line contained in $\mathcal{Q}(4,q)$, we will say that they are *collinear* (in $\mathcal{Q}(4,q)$); finally, every line ℓ not contained in $\mathcal{Q}(4,q)$ intersects $\mathcal{Q}(4,q)$ in $0,1$, or 2 points, and so ℓ is called *external*, *tangent*, or *secant*, respectively. For more details about polarities, see [**10**].

We proceed as in the article [**6**], but with certain variations. From now on, we assume q to be odd. Let $\mathcal{Q}^-(3,q)$ be an elliptic quadric of $\mathcal{Q}(4,q)$ contained in a hyperplane Σ of $\mathrm{PG}(4,q)$; no line ℓ of $\mathcal{Q}(4,q)$ can be contained in Σ since $\mathcal{Q}^-(3,q)$ does not contain lines, so ℓ intersects $\mathcal{Q}^-(3,q)$ in exactly one point. Hence, $\mathcal{Q}^-(3,q)$ is an ovoid of $\mathcal{Q}(4,q)$ and it is also called the *classical ovoid* of $\mathcal{Q}(4,q)$. Let now π be a plane of Σ that intersects $\mathcal{Q}^-(3,q)$ in a conic; the line $\pi^\perp$ can be either secant or external; in the first case we call the plane π *good*, in the second case *bad*. For more information we refer to [**4**]. If π is a good plane and we delete the points of $\pi \cap \mathcal{Q}^-(3,q)$ from $\mathcal{Q}^-(3,q)$ and add the points of $\pi^\perp \cap \mathcal{Q}(4,q)$, we obtain a set Θ of size q^2-q+2. If $P \in \pi^\perp$ is a singular point, then $P^\perp \cap \Sigma = \pi$, hence if a line $\ell \subset \mathcal{Q}(4,q)$ intersects $\pi^\perp$ in P, then ℓ intersects Σ in a point of π, so Θ is a partial ovoid of $\mathcal{Q}(4,q)$. Moreover, if we add a point $R \notin \pi^\perp$ to Θ, then $R^\perp \cap \Sigma$ is a plane (different from π) containing a conic, so there would be lines of $\mathcal{Q}(4,q)$ with two points. Hence, we can conclude that Θ is a maximal partial ovoid of $\mathcal{Q}(4,q)$ of size q^2-q+2. In order to obtain a spectrum result for the size of Θ, we can delete the points of more conics of $\mathcal{Q}^-(3,q)$ contained in good planes π and replace them by the singular points of $\pi^\perp$. While doing this, we need to check that:

- Θ is a partial ovoid, that is, the points we add must not be collinear in $\mathcal{Q}(4,q)$,
- Θ is maximal,
- the planes π we use in this construction have a polar line $\pi^{\perp}$ which is a secant line, and

furthermore we need to compute the exact number of the singular points of the planes π we are using.

3. The construction

3.1. The parabolic quadric $\mathcal{Q}(4,q)$ considered.

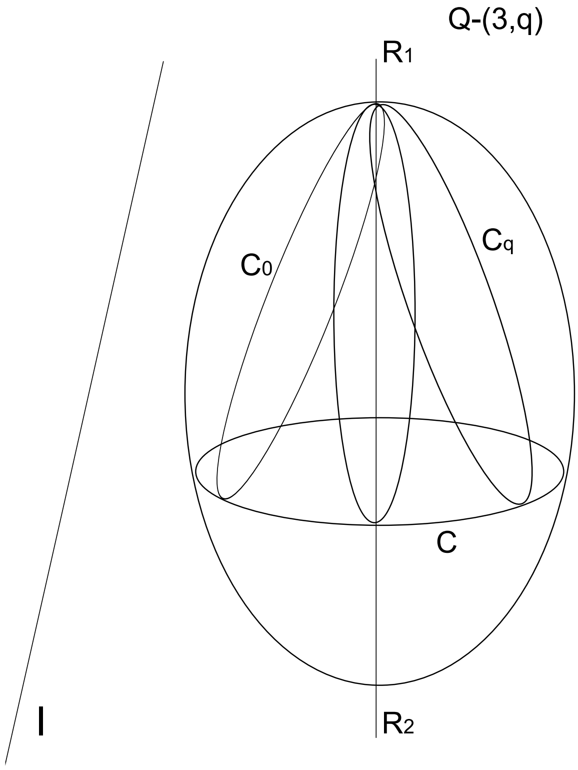

Figure 1: Set 1 of conics of $Q^{-}(3,q)$ in planes through ℓ and set 2 of conics

We first name the important elements involved in the construction. This includes: (1) the parabolic quadric $\mathcal{Q}(4,q)$, (2) in a particular hyperplane Σ, the elliptic quadric $\mathcal{Q}^{-}(3,q)$ contained in $\mathcal{Q}(4,q)$, (3) a fixed line ℓ in Σ skew to $\mathcal{Q}^{-}(3,q)$, and (4) the polar points R_1 and R_2 of ℓ with respect to $\mathcal{Q}^{-}(3,q)$. There is also a cyclic group C of order $q+1$ fixing R_1 and R_2, and stabilizing $\mathcal{Q}^{-}(3,q)$ which plays an important role in the construction of the maximal partial ovoids on $\mathcal{Q}(4,q)$.

Let $\{(x_0,x_1,x_2,x_3)||x_0 \in \mathbb{F}_{q^2}, x_1,x_2,x_3 \in \mathbb{F}_q\}$ be the underlying vector space of $\mathrm{PG}(4,q)$ and let

$$X_0^{q+1} + X_1X_2 + X_3^2 = 0$$

be the equation of the particular quadric $\mathcal{Q}(4,q)$. If $P=(a_0,a_1,a_2,a_3)$, then $P^\perp$ is the hyperplane with equation $Tr(a_0^q X_0)+a_2X_1+a_1X_2+2a_3X_3=0$, where Tr is the trace function from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$. The hyperplane Σ has equation $X_3=0$ and $\Sigma\cap\mathcal{Q}(4,q)$ is the elliptic quadric $\mathcal{Q}^-(3,q)$ with equation $X_0^{q+1}+X_1X_2=0$; the line $\ell=\{(x_0,0,0,0)||x_0\in\mathbb{F}_{q^2}\}$ is an external line contained in Σ and $\ell^\perp\cap\Sigma$ is a line intersecting $\mathcal{Q}^-(3,q)$ in two points: $R_1=(0,1,0,0)$ and $R_2=(0,0,1,0)$.

Let C be the set of the elements x of $\mathbb{F}_{q^2}$ such that $x^{q+1}=1$; C is a cyclic (multiplicative) group of order $q+1$ and let η be its generator. By abuse of notation, we denote by C also the cyclic group of collineations of $\mathrm{PG}(4,q)$ acting as follows:

$$(x_0,x_1,x_2,x_3)\longmapsto(\eta^i x_0,x_1,x_2,x_3).$$

The group C clearly fixes the quadrics $\mathcal{Q}(4,q)$ and $\mathcal{Q}^-(3,q)$, the line ℓ, and the points R_1 and R_2.

We assume that the cyclic group C of collineations of $\mathrm{PG}(4,q)$ described above is generated by the collineation α. For a given plane π in Σ, we denote its image under α^i by π^i. In particular, there is one involution in C, the transformation $\alpha^{(q+1)/2}:(x_0,x_1,x_2,x_3)\longmapsto(-x_0,x_1,x_2,x_3)$, and then $\pi^{(q+1)/2}$ is the image of the plane π under this involution.

The two sets of planes we want to use in our construction of maximal partial ovoids are all contained in the hyperplane Σ, hence we omit the equation $X_3=0$ each time and we just use the equations that describe them in $\Sigma: X_3=0$. Precisely, they are the following:

Set 1: The $q-1$ planes through ℓ intersecting $\mathcal{Q}^-(3,q)$ in a conic. Each of these planes π has equation: $X_1+aX_2=0$, with $a\neq 0$, and $\pi\cap\mathcal{Q}^-(3,q)=\{(x,-a,1,0)||x^{q+1}=a\}$. Every plane in this set is fixed by C and the points of such a conic form an orbit under the action of the group C. These planes do not intersect each other in singular points, of course.

Set 2: One orbit (of size $q+1$) under the action of C among the q^2-1 planes through R_1, but not through R_2, intersecting $\mathcal{Q}^-(3,q)$ in a conic. Such planes π have equation: $Tr(A\eta^i X_0)+X_2=0$, with $A\in\mathbb{F}_{q^2}\setminus\{0\}$ and $i=0,\ldots,q$.

3.2. Discussion of the intersection of the conics of set 1 with the conics in set 2. We are interested in how the conics of $\mathcal{Q}^-(3,q)$ in the planes of **set 2** intersect each other, that is whether two planes π_1 and π_2 in one orbit under the cyclic group C of order $q+1$ intersect in a secant or a tangent line. Applying the polarity induced by $\mathcal{Q}^-(3,q)$, this is equivalent to investigating whether the two polar points $\pi_1^\perp$ and $\pi_2^\perp$ w.r.t. $\mathcal{Q}^-(3,q)$ lying in the plane $R_1^\perp: X_2=0$ generate an external or a tangent line w.r.t. $\mathcal{Q}^-(3,q)$. If π_1 has equation $Tr(AX_0)+X_2=0$, then $\pi_1^\perp=(A^q,1,0,0)$ and the orbit of this point under C consists of the points $(A^q\eta^i,1,0,0)$, $i=1,\ldots,q+1$; this is the conic of the plane $R_1^\perp$ of equation $A^{q+1}X_1^2=X_0^{q+1}$. The only lines tangent to $\mathcal{Q}^-(3,q)$ in these planes are the ones through $R_1=(0,1,0,0)$ and the only tangent line through $\pi_1^\perp$ is the one joining $\pi_1^\perp=(A^q,1,0,0)$ to $(-A^q,1,0,0)$. Note that these two points are each others image under the involution $\alpha^{(q+1)/2}$.

Going back to the planes of **set 2**, this means that for every fixed plane π in this set, there exists only one plane in the same orbit of π under C that intersects π in a tangent line through R_1. From the preceding paragraph, it follows that π and its image under the involution $\alpha^{(q+1)/2}$ intersect in a tangent line to $\mathcal{Q}^-(3,q)$.

The other planes under the orbit of C intersect π in a secant line and, since any three points $\pi_1^\perp$, $\pi_2^\perp$, and $\pi_3^\perp$ are never collinear, the secant lines are all different, for every $\pi_i^\perp, i = 1,2,3$, in the same orbit under the cyclic group C.

We also need to investigate how two planes of the two different sets intersect each other. The line ℓ intersects a plane π of **set 2** in just one point, say P, and of course P is a non–singular point. Since the plane generated by ℓ and R_1 is tangent to $\mathcal{Q}^-(3,q)$, the line $\langle P, R_1\rangle$ is a tangent line with respect to the conic $\pi \cap \mathcal{Q}^-(3,q)$, hence there is exactly one other tangent line to $\pi \cap \mathcal{Q}^-(3,q)$ through P. This means that there is one other plane in **set 1** that intersects a plane of **set 2** in a tangent line, $\frac{q-1}{2}$ planes intersect in a secant line, and the remaining $\frac{q-1}{2}$ planes in an external line.

3.3. Determining the good planes in set 1 and finding a set 2 completely consisting of good planes. Let π be a plane in one of these two sets: we want to replace the singular points of π by the common singular polar points, i.e. by the singular points of $\pi^\perp$. Since q is odd, the plane $\pi^\perp$ with respect to the polarity induced by $\mathcal{Q}(4,q)$ can be either a secant or an external line and, of course, we need to avoid the latter case.

A plane π_1 in **set 1** in $\mathrm{PG}(4,q)$ has equations: $\begin{cases} X_1 + aX_2 = 0, \\ X_3 = 0, \end{cases}$ with $a \neq 0$, hence $\pi_1^\perp$ is the line $\langle(0,a,1,0),(0,0,0,1)\rangle$. It is easy to check that $\pi_1^\perp$ is a secant line if and only if $-a$ is a non-zero square in $\mathbb{F}_q$, hence there are $\frac{q-1}{2}$ good planes in **set 1**.

If π_2 is a plane of **set 2**, then it has as equations: $\begin{cases} Tr(AX_0) + X_2 = 0, \\ X_3 = 0, \end{cases}$ with $A \neq 0$, so $\pi_1^\perp$ is the line $\langle(A^q,1,0,0),(0,0,0,1)\rangle$ and this is a secant line if and only if $-A^{q+1}$ is a non-zero square in $\mathbb{F}_q$. Hence, in one orbit under C, the planes are all of the same type, so we can assume that in our case all the planes of **set 2** are good.

Moreover, for our construction, it is useful to know which planes in **set 1** that intersect the planes of **set 2** in a secant line are good. Again, we look at their polar points $(0,a,1,0)$ and $(A^q,1,0,0)$ w.r.t $\mathcal{Q}^-(3,q)$, and we have the line $\langle(0,a,1,0),(A^q,1,0,0)\rangle$. The two planes intersect in two singular points if and only if this polar line $\langle(0,a,1,0),(A^q,1,0,0)\rangle$ is external to $\mathcal{Q}^-(3,q)$. In our setting, this polar line is an external line if and only if $1-4aA^{q+1}$ is a non-square, a bisecant line if and only if $1-4aA^{q+1}$ is a non-zero square, and a tangent line if and only if $1-4aA^{q+1}$ is zero. In this last case, $a = 1/(4A^{q+1})$, so $-a = 1/(4(-A^{q+1}))$ is a non-zero square in $\mathbb{F}_q$ since $-A^{q+1}$ is a non-zero square in $\mathbb{F}_q$. We conclude that there is one good plane in **set 1** tangent to all the conics of **set 2**.

Since we consider an element A such that $-A^{q+1}$ is a non-zero square and since $-a$ is a non-zero square for a good plane in **set 1**, we first determine how many times $1-4(-a)(-A^{q+1})$ is a non-zero square. This is related to finding how many $x^2 \neq 0$ satisfy the equation $1-x^2 = y^2$. This is the equation of an affine conic that has two points at infinity if -1 is a square, i.e. $q \equiv 1 \bmod 4$, or none otherwise, when $q \equiv 3 \bmod 4$. There are always two points corresponding to $y = 0$ and there are always two points corresponding to the value $x = 0$, so there are $\frac{q-5}{4}$ (resp. $\frac{q-3}{4}$) values of $x^2 \neq 0$ for $q \equiv 1 \bmod 4$ (resp. for $q \equiv 3 \bmod 4$) that satisfy the equation $1-x^2 = y^2$. Hence, among the $(q-1)/2$ good planes in **set 1**, there is one tangent to the conics of **set 2** and $(q-5)/4$ skew to the conics of **set 2**. More precisely, if $a = \frac{1}{4A^{q+1}}$, then the corresponding good plane in **set 1** intersects all the planes in **set 2** in a tangent line and there are $\frac{q-1}{2} - 1 - \frac{q-5}{4} = \frac{q-1}{4}$ (resp. $\frac{q-3}{4}$) good planes in **set 1** that intersect the planes of **set 2** in two singular points if $q \equiv 1 \bmod 4$ (resp. if $q \equiv 3 \bmod 4$).

We summarize the results of this paragraph in the following lemma.

LEMMA 3.1. *There is one good plane of* ***set 1*** *that intersects all the planes in* ***set 2*** *in a tangent line and there are* $\frac{q-1}{4}$ *(resp.* $\frac{q-3}{4}$*) good planes in* ***set 1*** *that intersect the planes of* ***set 2*** *in a secant line if* $q \equiv 1 \bmod 4$ *(resp. if* $q \equiv 3 \bmod 4$*).*

3.4. Replacing deleted good conics of $\mathcal{Q}^-(3,q)$ by their polar points. When we replace the points of a good conic of $\mathcal{Q}^-(3,q)$ by their common polar points, we need to check that the new set is still a partial ovoid, meaning the points added are not collinear in $\mathcal{Q}(4,q)$ among themselves and with the other points of the partial ovoid.

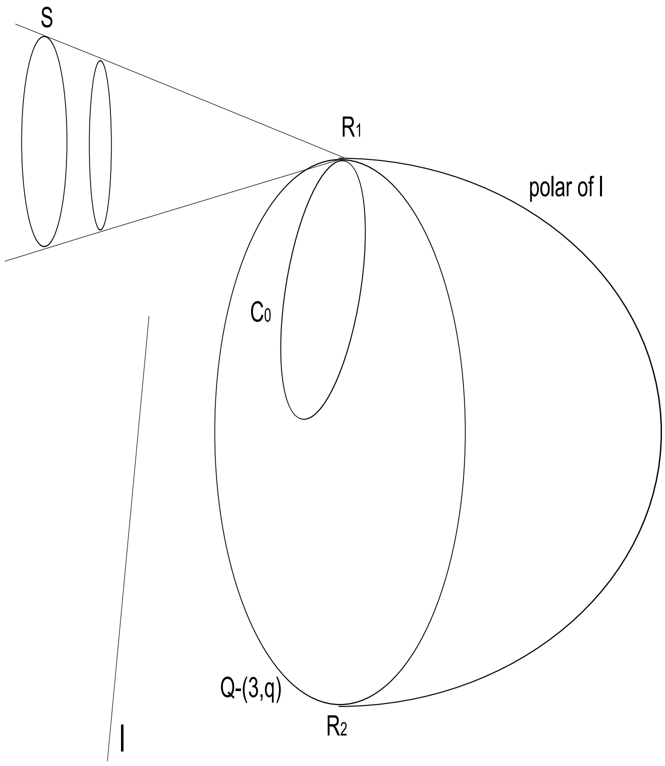

Figure 2: Polar points of the good conics in **set 1** and of the conics in **set 2**

For every point $P \notin \Sigma$ that we add, we know that $P^{\perp} \cap \mathcal{Q}^-(3,q)$ is a conic that we have thrown away, so none of its lines of $\mathcal{Q}(4,q)$ is collinear with a point still in Σ. We need to make sure that the points out of Σ which we add are not collinear with each other since we want to construct a new partial ovoid on $\mathcal{Q}(4,q)$. The polar lines of the good planes of **set 1** are the lines through the point $\Sigma^{\perp}$ in the plane $\ell^{\perp}$ secant to the conic of $\mathcal{Q}(4,q)$ contained in $\ell^{\perp}$; of course they are two by two not collinear. The polar lines of the planes in **set 2** are the lines in $R_1^{\perp}$ through the point $\Sigma^{\perp}$ and they are secant to the tangent cone contained in $R_1^{\perp}$. Using coordinates, the points of intersection are: $(\eta^i A^q, 1, 0, \pm\sqrt{-A^{q+1}})$, where $\sqrt{a}$ is one of the two elements in $\mathbb{F}_q$ whose square is a. They form two conics in the hyperplane $R_1^\perp : X_2 = 0$, one in the plane $X_3 = \sqrt{-A^{q+1}}X_1$ and the other one in the plane $X_3 = -\sqrt{-A^{q+1}}X_1$; a point $(\eta^i A^q, 1, 0, \sqrt{-A^{q+1}})$ of the first conic is collinear on $\mathcal{Q}(4,q)$ with the point $(-\eta^i A^q, 1, 0, -\sqrt{-A^{q+1}})$, which means the polar points of the conic of the plane $\pi \in$ **set 2** are collinear with a polar point of the conic of $\pi^{(q+1)/2}$, where $\pi^{(q+1)/2}$ is the image of π under the collineation of C of order two. Let now $(0, a, 1, \sqrt{-a})$ be one of the polar points added in place of a good plane of **set 1** with equations $\begin{cases} X_1 + aX_2 = 0, \\ X_3 = 0, \end{cases}$; it is collinear with $(\eta^i A^q, 1, 0, \pm\sqrt{-A^{q+1}})$ if and only if $a = \frac{1}{4A^{q+1}}$, thus when the plane in **set 1** intersects the planes of **set 2** in a tangent line.

3.5. Constraints on the parameters involved. We now have to find the constraints on the parameters that are required to obtain a non-interrupted interval of sizes k for maximal partial ovoids on $\mathcal{Q}(4,q)$.

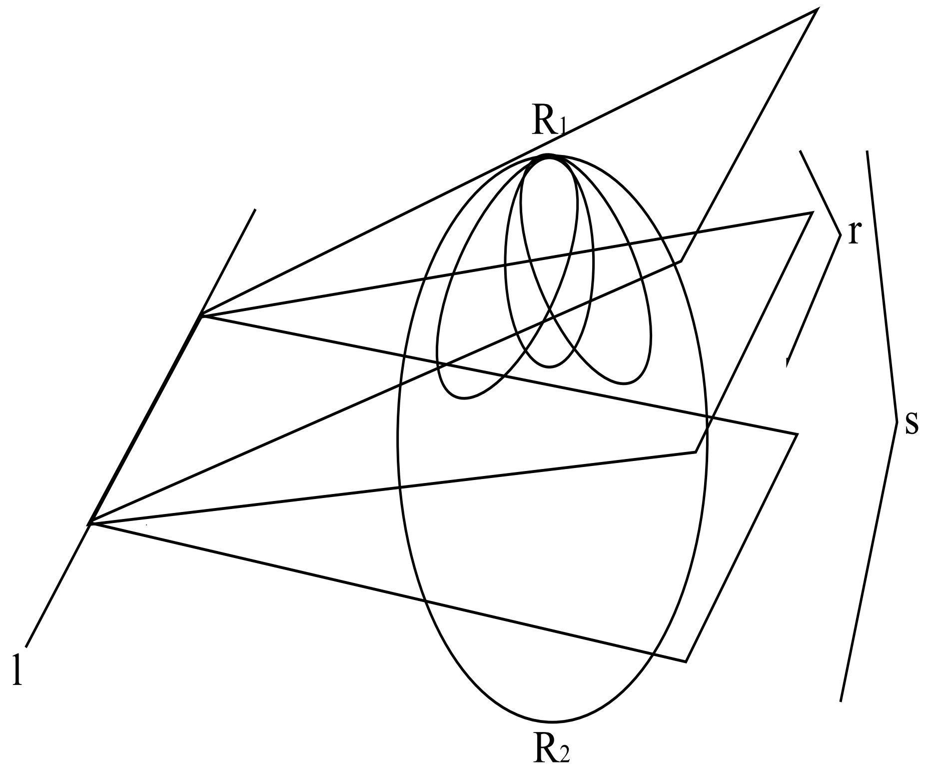

Figure 3: The parameters r and s involved in the construction

Let s be the number of planes of **set 1** that we do not replace, let t be the number of planes in **set 2** that we replace, let r be the number of planes in **set 1** that we do not replace and that intersect the planes in **set 2** in a secant line, and let u be the number of points, different from R_1, in which the conics in the planes of **set 2** thrown away intersect each other. We have indicated these s and r planes of **set 1** in Figure 3. In order to get a partial ovoid after the replacement, we need to impose:

$$1 \leq t \leq \frac{q+1}{2}$$

since **set 2** consists of one orbit of good planes under the action of C, but in order to avoid collinear polar points, if we replace the points of the plane π in **set 2**, we can not replace the points of $\pi^{(q+1)/2}$, since they have collinear polar points on $\mathcal{Q}(4,q)$ (Subsection 3.4), hence we can replace at most the points of the planes $\pi^i, i = 1, \ldots, \frac{q+1}{2}$.

Moreover, we have

$$\frac{q+1}{2} \leq s \leq q-1,$$

because there are $\frac{q-1}{2}$ bad planes in **set 1**, which we do not replace, and there is also the good plane through ℓ that intersects the planes of **set 2** in a tangent line, hence the polar points added would be collinear if we would throw away this plane, so we can not replace the points of at least $\frac{q+1}{2}$ planes through ℓ.

In this way, we know that the newly constructed set Θ is a partial ovoid of $\mathcal{Q}(4,q)$, but Θ has to be also maximal, hence for every point P of $\mathcal{Q}(4,q) \setminus \Theta$, there exists at least one point of $P^\perp$ in Θ. This is of course true for every point of Σ; let P be a point of $\mathcal{Q}(4,q)$ not in Σ and let π_P be the plane $P^\perp \cap \Sigma$: we impose that $\pi_P \cap \Theta \neq \emptyset$. We have different cases:

1) $\ell \subseteq \pi_P$: the other planes of **set 1** do not intersect π_P in any singular point, while the planes of **set 2** can intersect π_P in at most two points, hence we impose that $t < \frac{q+1}{2}$ to make sure that $\pi_P \cap \Theta \neq \emptyset$.
2) $\ell \not\subseteq \pi_P$ and $R_2 \in \pi_P$: R_2 is not contained in any of the conics of $\mathcal{Q}^-(3,q)$ that we throw away, so π_P contains always at least the point R_2 of Θ.
3) $\ell \not\subseteq \pi_P$, $R_1 \in \pi_P$ and $R_2 \notin \pi_P$; in this case we have two subcases:
   3.a) π_P is one of the planes of **set 2**: this plane is tangent to a particular plane π of **set 1** in one singular point P'; the conic of the plane π is not deleted (see the condition for s above) and the other planes of **set 2** intersect π_P in points different from P', hence we know that the point P' is never thrown away from π_P.
   3.b) π_P is in another orbit under the action of C. Let π_P: $Tr(A'X_0) + X_2 = 0$. Checking the four distinct cases for $(-A'^{q+1}, q)$, $-A'^{q+1}$ is a non–zero square or a non–square, $q \equiv 1 \bmod 4$ or $q \equiv 3 \bmod 4$, if we impose $t < \frac{q-1}{2}$, then the good planes of **set 1** and the t deleted conics of **set 2** cannot cover all the points of the conic of π_P.
4) $\ell \not\subseteq \pi_P$ and R_1 and $R_2 \notin \pi_P$: the planes of **set 1** intersect π_P in at most 2 singular points. We consider the following two cyclic groups fixing the line ℓ: the group C_1 of size $\frac{q-1}{2}$ that acts regularly on the good planes through ℓ, and the group C_2 that acts regularly on the planes through ℓ that intersect π_P in a secant line, so C_2 has size $\frac{q+1}{2}$ or $\frac{q-1}{2}$, according to the fact that through ℓ there are zero or two planes intersecting π_P in a tangent line. Since these two groups fix a line in a three–dimensional space, we can assume that C_1 and C_2 are subgroups of $\mathrm{PGL}(2,q)$, so we can use Theorem 3.4, Theorem 3.5, and Corollary 3.6 of [**8**] and state that the good planes of **set 1** intersecting π_P in a secant line are at most $\frac{q+3\sqrt{q}}{4}$. In order to keep at least one of the points of the conic of π_P in

Θ, we need to impose $2t+2+\frac{q+3\sqrt{q}}{2}<q+1$, where $2t$ comes from the at most $2t$ intersection points of the t deleted conics of **set 2** with π_P, and where 2 comes from the at most two tangent points of the two possible good planes through ℓ tangent to π_P.

To conclude, we have the following new constraint for t:

$$t<\frac{q-3\sqrt{q}-2}{4}.$$

Finally, for the parameter r we have:

a) $\frac{q-1}{4}\leq r\leq s-\frac{q-1}{4}-1$ for $q\equiv 1 \bmod 4$, and $\frac{q+1}{4}\leq r\leq s-\frac{q-3}{4}-1$ for $q\equiv 3 \bmod 4$,
b) $s=\frac{3}{4}(q-1)\Rightarrow\frac{q-1}{4}\leq r\leq\frac{q-1}{2}-1$ for $q\equiv 1 \bmod 4$, and $s=\frac{3q-1}{4}\Rightarrow\frac{q+1}{4}\leq r\leq\frac{q-1}{2}$ for $q\equiv 3 \bmod 4$,
c) $s>\frac{3}{4}(q-1)$ for $q\equiv 1 \bmod 4$, and $s>\frac{3q-1}{4}$ for $q\equiv 3 \bmod 4\Rightarrow s-\frac{q-1}{2}\leq r\leq\frac{q-1}{2}$.

We give a brief explanation for the case $q\equiv 1 \bmod 4$. We know that there are $\frac{q-1}{4}$ (resp. $\frac{q+1}{4}$ for $q\equiv 3 \bmod 4$) bad planes in **set 1** that intersect the planes of **set 2** in a secant line (Lemma 3.1). These bad planes are never thrown away, hence we always have $\frac{q-1}{4}\leq r\leq\frac{q-1}{2}$. But the parameter r also depends on s. More precisely, these r planes are a subset of the s planes in **set 1** we have not replaced and among them we know that there is one good plane that intersects the planes of **set 2** in a tangent line (Lemma 3.1) and there are $\frac{q-1}{4}$ (resp. $\frac{q-3}{4}$) bad planes that intersect the planes of **set 2** in an external line (resp. for $q\equiv 3 \bmod 4$), hence we have $r\leq s-\frac{q-1}{4}-1$ (resp. $r\leq s-\frac{q-3}{4}-1$ for $q\equiv 3 \bmod 4$). Finally, from a certain value for s, as the value of s increases, also r does. There are $(q-3)/2$ conic planes in **set 1** skew to the conics of **set 2**, and there is one good plane in **set 1** tangent to all the conics of **set 2**. Hence, if $s-\frac{q-1}{2}$ is larger than the number of bad planes in **set 1** that intersect planes in **set 2** in a secant line, then $r\geq s-\frac{q-1}{2}$, hence $s>\frac{3}{4}(q-1)\Rightarrow s-\frac{q-1}{2}\leq r$. The constraints mentioned before arise just by the comparison of these upper and lower bounds.

For every fixed s,t,r, and u, we get that the size of the maximal partial ovoid Θ on $\mathcal{Q}(4,q)$ is $s(q-1)+2q-2tr+t+u-1$. This is proven in the following way.

We do not replace s of the conics of **set 1**; equivalently, we replace $q-1-s$ of the conics of **set 1** by their two polar points. This changes the size of the ovoid, i.e. the elliptic quadric $\mathcal{Q}^-(3,q)$, from q^2+1 to $q^2+1-(q-1-s)(q+1)+2(q-1-s)=s(q-1)+2q$. We then delete the points of t conics in **set 2**. But some of the points of these t conics are already deleted. There are r conics in **set 1** that are not deleted and that intersect the conics of **set 2** in two points. There is one good plane in **set 1** tangent to all the conics of **set 2**, and also this conic is not deleted. Also the point R_1 belonging to all the conics in **set 2** has not yet been deleted. So $2r+1+1$ points in every conic of **set 2** still belong to the already constructed set of size $s(q-1)+2q$. The t conics in **set 2** that will be deleted, and replaced by their polar points, intersect, by assumption, in u points, different from R_1. So we only delete $t(2r+1)+1-u$ points from these t conics of **set 2**, and these $2tr+t-u+1$ points are replaced by the $2t$ polar points of these t conics of **set 2**. Hence, the size of the newly constructed set Θ is

$$s(q-1)+2q-2tr-1-t+u+2t = s(q-1)+2q-2tr+t+u-1.$$

3.6. Selection of $t = 5$ conics of set 2. In order to get a non-interrupted interval of values for the size of Θ, we proceed as in [**6**], i.e. we select five planes in **set 2** such that their ten intersection points (different from R_1) are partitioned in four planes of **set 1**, namely $\pi_i, i = 1,\dots,4$, contains i of these points. In this way, we set $t = 5$ and choosing the planes in **set 1** in a suitable way, we can let the parameter u vary from 0 to 10, so we immediately get the first non-interrupted interval

$$[(s+2)q-s+4-10r, (s+2)q-s+14-10r].$$

As in [**6**], we consider a plane π in **set 2** and the planes $\pi^i, i = 1,\dots,4$, which are the image of π under α, where α is a generator of the cyclic group C; the intersection points are the following: $\pi \cap \pi^i = P_i, i = 1,\dots,4$, $\pi^1 \cap \pi^i = P^1_{i-1}, i = 2,3,4$, $\pi^2 \cap \pi^i = P^2_{i-2}, i = 3,4$, and $\pi^3 \cap \pi^4 = P^3_1$ (here, similarly, P^k_j denotes the image of the point P_j under α^k. In [**6**], it is proved that, for $q > 5$, there exists a plane π_1 in **set 1** that contains the points $P_1, P^i_1, i = 1,2,3$, a plane π_2 of **set 1** that contains $P_2, P^i_2, i = 1,2$, a plane π_3 of **set 1** that contains P_3 and P^1_3, and there is a plane π_4 in **set 1** that contains only the intersection point P_4. The main difference here, in comparison to [**6**], is that we have to check that the four planes $\pi_i, i = 1,\dots,4$, in **set 1** with the required property are good planes. If the plane π has equation $Tr(AX_0) + X_2 = 0$, then the intersection $\pi \cap \mathcal{Q}^-(3,q)$ is the conic $\{(A^q + A\eta^i, 1, -2A^{q+1} - Tr(A^2\eta^i), 0) || \eta^i \in C\} = \{(A^q + A^q\eta^j, 1, -2A^{q+1} - A^{q+1}Tr(\eta^j), 0) || \eta^j \in C\}$. There is one plane in **set 2** that intersects π in a tangent line through R_1. Every point of this conic in π, different from R_1 and different from one other particular point which is the intersection of π with the unique good plane of **set 1** tangent to the conic $\pi \cap \mathrm{Q}^-(3,q)$, is contained in just one other plane of **set 2**, namely the point $(A^q + A\eta^i, 1, -2A^{q+1} - Tr(A^2\eta^i), 0), \eta^i \neq A^{q-1}$, is contained in the plane with equation $Tr(\eta^{-i}A^qX_0) + X_2 = 0$ since we have that $Tr(\eta^{-i}A^q(A^q + A\eta^i)) = Tr(A(A^q + A\eta^i)) = 2A^{q+1} + Tr(A^2\eta^i)$. The unique good plane of **set 1** that is tangent to all the conics of **set 2** is the plane $X_1 + aX_2 = 0$, with $a = \frac{1}{4A^{q+1}}$ (See Subsection 3.3). This contains the point $(A^q + A\eta^i, 1, -2A^{q+1} - Tr(A^2\eta^i), 0), \eta^i = A^{q-1}$, so the point $(2A^q, 1, -4A^{q+1}, 0)$. The singular point in $\pi \cap \pi^{-j}$, $j \neq (q+1)/2$, different from R_1, is the point $P_j = (A^q + A^q\eta^{-j}, 1, -2A^{q+1} - A^{q+1}Tr(\eta^j))$. The point P_j is contained in the plane of **set 1** with equation $X_1 + aX_2 = 0$, with $a = \frac{1}{A^{q+1}(2+Tr(\eta^j))} = \frac{1}{A^{q+1}(1+\eta^j)^{q+1}}$. As we want these planes to be good, we need $-a$ to be a non-zero square, that is $(1+\eta^j)^{q+1}$ is a non-zero square since $-A^{q+1}$ is a non–zero square, and this happens if and only if $1 + \eta^j$ is a non-zero square in $\mathbb{F}_{q^2}$. So we need to prove the following lemma.

Lemma 3.2. *Let $C = \langle \eta \rangle$ be the cyclic multiplicative group of the $(q+1)$-th roots of unity in the field of odd characteristic $\mathbb{F}_{q^2}$, then $1 + \eta^i$, $\eta^i \in C \setminus \mathbb{F}_q$, is a non-zero square in $\mathbb{F}_{q^2}$ if and only if i is even.*

Proof. If we have $1 + \eta^{2i}$, then $1 + \eta^{2i} = \eta^i(\frac{1}{\eta^i} + \eta^i) = \eta^i Tr(\eta^i)$; if ξ is a primitive element of $\mathbb{F}_{q^2}$, then we can say that $\eta = \xi^{q-1}$ and that a primitive element of $\mathbb{F}_q$ is ξ^{q+1}, so every element of C and every element of $\mathbb{F}_q$ are squares in $\mathbb{F}_{q^2}$, so $\eta^i Tr(\eta^i)$ is a square too.

Viceversa, if we have that $1+\eta^i = d^2$, for some non-zero $d \in \mathbb{F}_{q^2}$, then $1+\eta^i = d^2 \Rightarrow \eta^i = d^2-1 \Rightarrow 1 = (d^2-1)^{q+1} \Rightarrow 1 = d^{2(q+1)}+1-d^2-d^{2q} \Rightarrow d^{2(q+1)} = d^2+d^{2q} \Rightarrow d^{2q} = 1+d^{2(q-1)} \Rightarrow d^2 = 1+d^{2(1-q)} = 1+\eta^i$. Consequently, $d^{2(1-q)} = \eta^i$ and hence i has to be even. $\square$

So the planes to use are $\pi, \pi^2, \pi^4, \pi^6, \pi^8$, more precisely, for a given plane π of **set 2**, π has intersection points $\pi \cap \pi^{2i} = P_{2i}, i = 1, \dots, 4$, $\pi^2 \cap \pi^{2i} = P^2_{2(i-1)}, i = 2, \dots, 4$, $\pi^4 \cap \pi^{2i} = P^4_{2(i-2)}, i = 3, 4$, and $\pi^6 \cap \pi^8 = P^6_2$.

3.7. Determination of the non-interrupted interval for the sizes of maximal partial ovoids on $\mathcal{Q}(4,q)$. Now we can calculate the non-interrupted interval of values for the size of Θ. The case $q \equiv 1 \bmod 4$ and the case $q \equiv 3 \bmod 4$ need to be treated separately, but we omit the proof for the case $q \equiv 3 \bmod 4$.

Maintaining the same notations as before, we know that for a fixed value of s and r, and for $t = 5$, choosing the planes in **set 1** in a suitable way, i.e. using the planes $\pi_i, i = 1, \dots, 4$, of **set 1**, in the way described above, we can let vary the parameter u from 0 to 10, so we immediately get the first non-interrupted interval

$$[(s+2)q - s + 4 - 10r, (s+2)q - s + 14 - 10r]. \tag{3.1}$$

But we now have slightly different constraints for the parameters, since we need a certain freedom to take or not take the four planes $\pi_i, i = 1, \dots, 4$, so for $q \equiv 1 \bmod 4$, we have:

a) $\frac{q+1}{2} + 4 \leq s \leq q - 5$,
b) if $s \leq \frac{3(q-1)}{4} - 3$, then $\frac{q-1}{4} + 4 \leq r \leq s - \frac{q-1}{4} - 1$,
c) if $\frac{3(q-1)}{4} - 2 \leq s \leq \frac{3(q-1)}{4} + 3$, then $\frac{q-1}{4} + 4 \leq r \leq \frac{q-1}{2} - 4$,
d) if $s \geq \frac{3(q-1)}{4} + 4$, then $s - \frac{q-1}{2} \leq r \leq \frac{q-1}{2} - 4$.

The interval (3.1) has size 10, so if we let vary the parameter r and we fix s, we still get a non-interrupted interval. Taking into consideration condition **b**), we have the interval

$$[s(q-11) + \frac{9q+23}{2}, s(q-1) - \frac{q+47}{2}], \tag{3.2}$$

if $s \leq \frac{3(q-1)}{4} - 3$, but if we want to let vary s too, we need to impose that $s'(q-11) + \frac{9q+23}{2} \leq s(q-1) - \frac{q+47}{2}$, where $s' = s+1$. By straightforward calculations, we get $\frac{3q+12}{5} \leq s$. Letting vary s in $[\frac{3q+12}{5}, \frac{3(q-1)}{4} - 3]$, then from (3.2), we get the interval:

$$[\frac{6q^2+3q-149}{10}, \frac{3q^2-20q-79}{4}]. \tag{3.3}$$

From the condition **c**), we have the interval:

$$[s(q-1) - 3q + 49, s(q-1) - \frac{q+47}{2}] \tag{3.4}$$

and the size of this interval is $\frac{5q-145}{2} \geq q-1$ if $q \geq \frac{143}{3}$, so we can let vary s from $\frac{3(q-1)}{4}-2$ to $\frac{3(q-1)}{4}+3$ in (3.4) and obtain the interval:

$$[\frac{3q^2-26q+207}{4}, \frac{3q^2+4q-103}{4}]. \tag{3.5}$$

Finally, from **d)**, we have the interval:

$$[s(q-1)-3q+49, s(q-11)+7q+9] \tag{3.6}$$

and to let vary s, we need to impose that $s'(q-1)-3q+49 \leq s(q-11)+7q+9$, where $s'=s+1$. In this way, we get $s \leq \frac{9q-39}{10}$ and so we let vary s in $[\frac{3(q-1)}{4}+4, \frac{9q-39}{10}]$ in (3.6), and so we obtain the interval:

$$[\frac{3q^2-2q+183}{4}, \frac{9q^2-68q+519}{10}]. \tag{3.7}$$

It is clear that the intervals (3.3), (3.5), and (3.7) overlap if $q \geq \frac{143}{3}$ and so we have proven the following result.

THEOREM 3.3. *For every integer k in the interval $[\frac{6q^2+3q-149}{10}, \frac{9q^2-68q+519}{10}]$, there exists a maximal partial ovoid Θ of $\mathcal{Q}(4,q)$, $q \geq 49$ odd and $q \equiv 1$ mod 4, such that $|\Theta| = k$.*

When $q \equiv 3$ mod 4, we use exactly the same arguments, with the difference that the constraints for the parameters in this case are:

a) $\frac{q+1}{2}+4 \leq s \leq q-5$,
b) if $s \leq \frac{3q-1}{4}-4$, then $\frac{q+1}{4}+4 \leq r \leq s-\frac{q-3}{4}-1$,
c) if $\frac{3q-1}{4}-3 \leq s \leq \frac{3q-1}{4}+3$, then $\frac{q+1}{4}+4 \leq r \leq \frac{q-1}{2}-4$,
d) if $s \geq \frac{3q-1}{4}+4$, then $s-\frac{q-1}{2} \leq r \leq \frac{q-1}{2}-4$,

and this leads to the following same non–interrupted interval as for $q \equiv 1$ mod 4.

THEOREM 3.4. *For every integer k in the interval $[\frac{6q^2+3q-149}{10}, \frac{9q^2-68q+519}{10}]$, there exists a maximal partial ovoid Θ of $\mathcal{Q}(4,q)$, $q \geq 51$ odd and $q \equiv 3$ mod 4, such that $|\Theta| = k$.*

A *spread* $\mathcal{S}$ of a generalized quadrangle Γ is a set of lines of Γ such that every point of Γ is contained in exactly one line of $\mathcal{S}$; a *partial spread* $\mathcal{S}$ of Γ is a set of lines of Γ such that every point of Γ is contained in at most one line of $\mathcal{S}$. A partial spread $\mathcal{S}$ of Γ is called *maximal* when it is not contained in a larger partial spread of Γ. It is clear that in the dual generalized quadrangle, $\mathcal{S}$ corresponds to a partial ovoid. So, since $\mathcal{W}(q)$, q odd, is dual to $\mathcal{Q}(4,q)$, q odd, it is clear that from the last two theorems, we immediately have the following result.

COROLLARY 3.5. *For every integer k in the interval $[\frac{6q^2+3q-149}{10}, \frac{9q^2-68q+519}{10}]$, there exists a maximal partial spread $\mathcal{S}$ of the generalized quadrangle $\mathcal{W}(q)$, $q \geq 49$ odd, of size k.*

4. Concluding tables

As indicated in the abstract, this article concludes a series of three articles on spectrum results.

The initial article focussed on a spectrum result on maximal partial ovoids of $\mathcal{Q}(4,q)$, q even. Since $\mathcal{Q}(4,q)$ is dual to the generalized quadrangle $\mathcal{W}(q)$, the same spectrum result for maximal partial spreads of $\mathcal{W}(q)$, q even, is obtained.

Similarly, since $\mathcal{Q}(4,q)$, q even, (and $\mathcal{W}(q)$, q even), is self-dual, the same spectrum result on maximal partial spreads of $\mathcal{Q}(4,q)$, q even, and on maximal partial ovoids of $\mathcal{W}(q)$, q even, is obtained.

Moreover, a maximal partial ovoid of $\mathcal{W}(q)$, q even, is a minimal blocking set with respect to the planes of $\mathrm{PG}(3,q)$, q even [**2**], so also for these minimal blocking sets and, as a equivalent result, for maximal partial 1–system of the Klein quadric $\mathcal{Q}^+(5,q)$, the same spectrum result is obtained.

In these results, the following cases are not yet discussed: a spectrum result on minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$, q odd, and the equivalent spectrum results on maximal partial ovoids of $\mathcal{Q}(4,q)$, q odd, and on maximal partial spreads of $\mathcal{W}(q)$, q odd.

The article [**7**] presented a spectrum result on minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$, q odd, and this article discusses the spectrum results on maximal partial ovoids of $\mathcal{Q}(4,q)$, q odd, and on maximal partial spreads of $\mathcal{W}(q)$, q odd.

Finally, a minimal blocking set B with respect to the planes of $\mathrm{PG}(3,q)$ defines via the Klein correspondence a partial 1-system on the Klein quadric $\mathcal{Q}^+(5,q)$ [**6**].

A *1-system* $\mathcal{M}$ on $\mathcal{Q}^+(5,q)$ is a set of q^2+1 lines $\ell_1,\ldots,\ell_{q^2+1}$ on $\mathcal{Q}^+(5,q)$ such that $\ell_i^\perp \cap \ell_j = \emptyset$, for all $i,j \in \{1,\ldots,q^2+1\}$, $i \neq j$. A *partial 1-system* $\mathcal{M}$ on $\mathcal{Q}^+(5,q)$ is a set of $s \leq q^2+1$ lines $\ell_1,\ldots,\ell_s$ on $\mathcal{Q}^+(5,q)$ such that $\ell_i^\perp \cap \ell_j = \emptyset$, for all $i,j \in \{1,\ldots,s\}$, $i \neq j$. A partial 1-system on $\mathcal{Q}^+(5,q)$ is called *maximal* when it is not contained in a larger partial 1-system of $\mathcal{Q}^+(5,q)$.

To summarize these series of spectrum results, we gather the spectrum results in Tables 1, 2, and 3.

$\mathcal{W}(q)$, $\mathcal{Q}(4,q)$	Interval	
$q=2^{4h}$, $h\geq 2$	$\frac{q^2+194q+10q\lfloor 48\log(q+1)\rfloor-190}{10} \leq k \leq \frac{9q^2-69q+440}{10}$	[**6**]
$q=2^{4h+1}$, $h\geq 2$	$\frac{q^2+198q+10q\lfloor 48\log(q+1)\rfloor-230}{10} \leq k \leq \frac{9q^2-68q+430}{10}$	[**6**]
$q=2^{4h+2}$, $h\geq 1$	$\frac{q^2+196q+10q\lfloor 48\log(q+1)\rfloor-210}{10} \leq k \leq \frac{9q^2-66q+410}{10}$	[**6**]
$q=2^{4h+3}$, $h\geq 1$	$\frac{q^2+192q+10q\lfloor 48\log(q+1)\rfloor-170}{10} \leq k \leq \frac{9q^2-67q+420}{10}$	[**6**]

Table 1: Spectrum on maximal partial ovoids and on maximal partial spreads in $\mathcal{Q}(4,q)$, q even, and in $\mathcal{W}(q)$, q even, and on minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$, q even

$\mathrm{PG}(3,q)$	Interval	
$q\equiv 1 \pmod 4$	$k\in[(q^2+30q-47)/4+18(q-1)\log(q),(3q^2-18q+71)/4]$	[**7**]
$q\equiv 3 \pmod 4$	$k\in[(q^2+28q-37)/4+18(q-1)\log(q),(3q^2-12q+57)/4]$	[**7**]

Table 2: Spectrum on minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$, q odd

Interval
$k \in [\frac{6q^2+3q-149}{10}, \frac{9q^2-68q+519}{10}]$

Table 3: Spectrum on maximal partial ovoids of $\mathcal{Q}(4,q)$, q odd, and on maximal partial spreads of $\mathcal{W}(q)$, q odd

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